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The Non-Central Chi-Square Chart with Double Sampling

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Abstract

In this article, we consider a non-central chi-square chart with double sampling (*DS* χ^2 chart) to control the process mean and variance. As in the case of Shewhart control charts, samples of fixed size are taken from the process at regular time intervals; however, the sampling is performed in two stages. Let X be the process quality variable being measured. During the first stage, one item of the sample is inspected; if its X value is close to the target value of the process mean, then the sampling is interrupted. Otherwise, the sampling goes on to the second stage, where the remaining items are inspected and a non-central chi-square statistic, say T, is computed taking into account all n items of the sample, that is, their X values. A signal is triggered when the sample point given by the T value falls above the upper control limit of the proposed chart. The *DS* χ^2 chart performs better than the joint $\bar{X}$ and R charts, except when there is a large change in the process mean. Furthermore, if the *DS* χ^2 chart is used for monitoring diameters, volumes, weights, etc., then the employment of appropriate devices, such as go-no-go gauges can reduce the effort to decide if the sampling should go to the second stage or not.

Keywords: non-central chi-square chart, double sampling, joint $\bar{X}$ and R charts

Introduction

The Standard Shewhart control chart has been widely used for process surveillance because of its operational simplicity. However, this operational simplicity, that is, taking

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samples of size *n* from the process every *h* hours and searching for an assignable cause only when a point falls outside the control limits, makes the Shewhart chart slow in detecting small to moderate changes in the process parameter being controlled.

Among various statistical devices, specially designed to detect process changes quickly, is the adaptive control chart. A control chart is considered adaptive if at least one of its design parameters (that is, the sample size, the sampling interval, the coefficient of the control limit) varies as a function of the process data. For example, if the sample size is variable, then the position of each sample point on the chart establishes the size of the next sample (see Costa, 1994, or Epprecht and Costa, 2001). When this point falls inside the control limits, but near of one of them, it is reasonable to tighten the control by increasing the size of the next sample. On the other hand, if the sample point falls near the central line, it is reasonable to relax the control by decreasing the size of the next sample. The idea of varying the sample size (VSS) in an adaptive fashion can be applied to all chart's design parameters, including *h*, the sampling interval (see Costa, 1999a, or Epprecht et al., 2003), even though the logistical problem associated with the lack of fixed sampling points makes the use of variable sampling intervals (VSI) awkward. The number of samples in any given time period will be a random variable, and the time points at which the samples are taken will be unpredictable. Reynolds (1996a,b), Costa (1998a), and Lin and Chou (2005) considered a modification of the VSI idea. In this modification, samples are always taken at some fixed, equally-spaced time points, but additional samples are allowed between these time points whenever there is some indication of a process change. The charts using this modification of the VSI idea are called *variable sampling interval with sampling at fixed times* (VSIFT) control charts.

The results from published articles on adaptive control charts (see for example, Reynolds et al., 1988, 1990; Runger and Pignatiello, 1991; Saccucci et al., 1992; Amin and Miller, 1993; Runger and Montgomery, 1993; Prabhu et al., 1993, 1994, 1997; Costa, 1994, 1997, 1999a; Park and Reynolds, 1994, 1999; Das et al., 1997; De Magalhães et al., 2001, 2002) demonstrate that the adoption of adaptive schemes instead of fixed schemes can bring considerable economic benefits once their use leads to a better trade-off between the time to detect a process disturbance and the sampling rate required to control the process.

Double sampling (or two stage sampling) procedures combined with control charts is other alternative that has been used to improve the performance of the traditional Shewhart charts. The control charts where the samplings are performed in two stages (see Croasdale, 1974; Daudin, 1992; Steiner, 1999; Costa, 2000; Costa and Rahim, 2004) are, usually, faster than the standard Shewhart control charts to detect small to moderate shifts in the parameter process being controlled, without increasing the sampling frequency. During the first stage, one or more items of the sample are inspected and, depending on the results, the sampling is either interrupted or it goes on to the second stage, where the remaining sample items are inspected.

In recent years, considerable attention has been devoted to joint charts for monitoring the process mean and variance. For instance, Costa and Rahim (2000); and Rahim and Costa (2000) developed economic models to the joint $\bar{X}$ and *R* charts. Gan (1995) considered the joint *EWMA* charts; Albin et al. (1997) studied an *X* and an *EWMA* chart for individual observations. Chen et al. (2001) combined two *EWMA* charts into one chart and showed that the new *EWMA* chart is effective in detecting both increases and decreases in the process mean and/or variance. In the adaptive case, Costa (1998b, 1999b), and De Magalhães and Moura Neto (2005) studied the joint $\bar{X}$ and *R* charts with variable parameters. Reynolds and Stoumbos (2001) have investigated three joint charts for monitoring the mean and the variance of a normal quality variable using individual observations and variable sampling intervals. From these studies, one can observe that the joint charts are not totally reliable in identifying the nature of the disturbance. For example, if the joint $\bar{X}$ and *R* charts are in use, and the $\bar{X}$ chart signals the presence of an assignable cause, then it should be investigated which process parameter the assignable cause is affecting due to the fact that the *X* chart is not only sensitive to a shift in the process mean but also is sensitive to an increase in the process variance.

We propose in this paper the use of the double-sampling procedure with a non-central chi-square statistic to control the process mean and variance. As in the case with Shewhart charts, samples of fixed size are taken from the process at regular time intervals; however, the sampling is performed in two stages. Let *X* be the process quality variable being measured. During the first stage, one item of the sample is inspected; if its *X* value is close to the target value of the process mean, then the sampling is interrupted. Otherwise, the sampling goes on to the second stage, where the remaining items are inspected and a non-central chi-square statistic, say *T*, is computed taking into account all *m* items of the sample, that is, their *X* values. A signal is triggered when the sample point given by the *T* value falls above the upper control limit of the proposed chart. The performance of the proposed chart (*DS* χ^2 chart) is better than the joint $\bar{X}$ and *R* charts, except when there is a large change in the process mean. Furthermore, if the *DS* χ^2 chart is used for monitoring diameters, volumes, weights, etc., then the employment of appropriate devices, such as go-no-go gauges can reduce the effort to decide if the sampling should go to the second stage or not.

The Properties of the Non-Central Chi-Square Chart with Double Sampling

Throughout this article, it is assumed that the non-central chi-square chart with double sampling (*DS* χ^2 chart) is employed to monitor a process whose quality characteristic of interest (say, *X*) is normally distributed with mean μ and variance σ^2. The process is considered to start with the mean and the variance on target ($\mu = \mu_0$; $\sigma^2 = \sigma_0^2$; in-control state), but at some random time in the future an assignable cause shifts the mean from μ_0 to $\mu_1 = \mu_0 \pm \delta\sigma_0$, $\delta > 0$ and/or increases the variance from σ_0^2 to $\sigma_1^2 = \gamma^2\sigma_0^2$, $\gamma > 1$. The objective of process monitoring is the detection of any assignable cause that shifts μ and/or σ.

When there is a change in σ, it is usually assumed that the primary interest is in detecting increases in σ, because an increase corresponds to deterioration in quality.

The $\bar{X}$ and *R* charts are the traditional control charts used to detect changes in μ and σ, respectively. When two charts are used concurrently, a signal would be given if either chart indicates a possible occurrence of an assignable cause. For different values of δ and γ, Table 1 provides the probabilities $P_{\bar{X}}$, P_R, and $P_{\bar{X}-R}$ for samples of size $n = 5$ and $\alpha = 0.0027$, the risk of a false alarme when the joint $\bar{X}$ and *R* charts are in use. When the joint charts produce a signal, $P_{\bar{X}}$ is the probability that the signal was given only by the $\bar{X}$ chart, P_R is the probability that the signal was given only by the *R* chart, and $P_{\bar{X}-R}$ is the probability that the signal was given by both. From Table 1, one can observe that $P_{\bar{X}-R}$ has always a low value, even when a shift in the mean is accompanied by an increase in the variance ($\delta \neq 0$ and $\gamma > 1$). So, the major contribution of the joint $\bar{X}$ and *R* charts is on the process change detection and not on the identification of the nature of the change, whether the change is on the mean and/or on the variance. In practice, the speed with which the control charts detect process changes seems to be more important than their ability in identifying the nature of the change. Under these circumstances, it is advantageous to consider a single chart based on only one statistic to simultaneously monitor the process mean and variance. Domangue and Patch (1991), Gan (1995), and Chen et al. (2001 and 2004) have already explored the idea of using single charts to control the two parameters of the process.

When the *non-central chi-square* chart with double sampling (*DS* χ^2 chart) is in use, samples of size $m = n_0 + 1$ are randomly chosen at regular time intervals. Let X_{ij}, i = 1,2,3,..., and $j = 1,2,\ldots,m$ be the measurements of the variable *X* arranged in groups of size $m>1$, with *i* indexing the group number. The samplings are performed in two stages. During the first

Table 1 – Values of $P_{\bar{X}}$, P_R, and $P_{\bar{X}-R}$ for the joint $\bar{X}$ and *R* charts ($n = 5$).

δ	γ	$P_{\bar{X}}$	P_R	$P_{\bar{X}-R}$	δ	γ	$P_{\bar{X}}$	P_R	$P_{\bar{X}-R}$
	1.00	.4996	.4997	.0007		1.00	.9883	.0104	.0013
0.0	1.30	.3556	.6356	.0088	1.0	1.30	.8771	.1012	.0218
	1.50	.3095	.6680	.0225		1.50	.7436	.2022	.0541
	2.00	.2413	.6760	.0827		2.00	.4478	.3987	.1535
	1.00	.9104	.0884	.0012		1.00	.9944	.0043	.0013
0.5	1.30	.6473	.3366	.0161	1.25	1.30	.9221	.0550	.0229
	1.50	.5018	.4617	.0365		1.50	.8143	.1264	.0593
	2.00	.3084	.5859	.1057		2.00	.5152	.3081	.1767
	1.00	.9702	.0285	.0013		1.00	.9968	.0019	.0013
0.75	1.30	.7929	.1874	.0197	1.50	1.30	.9463	.0302	.0235
	1.50	.6387	.3148	.0465		1.50	.8596	.0778	.0626
	2.00	.3756	.4956	.1288		2.00	.5730	.2305	.1965

stage, the first item of the *i*-th sample is inspected. If its value, say X_{i1}, is close to the target value μ_0 (that is, $|X_{i1}-\mu_0| < w\sigma_0$, $w > 0$), then the sampling is interrupted. Otherwise, the second stage is initialized. During the second stage, the remaining n_0 items are inspected and the non-central chi-square statistic is computed.

$$T_i = \sum_{j=1}^{m}\left(X_{ij} - \mu_0 + \xi_i\sigma_0\right)^2, \qquad i = 1,2,... \tag{1}$$

We define $\xi_i = d$ if $X_{i1} > \mu_0$; otherwise $\xi_i = -d$, where d is a positive constant. Note that T_i is computed taking into account all m items of the sample, that is, their X values, including X_{i1}, the quality characteristic value from the item inspected during the first stage. A signal is given at sample i if $|X_{i1} - \mu_0| > w\sigma_0$ and $T_i > k\sigma^2_0$, where k is the factor used in determining the upper control limit for the non-central chi-square chart. During the in-control period, T_i/σ^2_0 is distributed as a non-central chi-square distribution with m degrees of freedom and a non-centrality parameter $\lambda_0 = nd^2$, i.e. $T_i/\sigma^2_0 \approx \chi^2_n(\lambda_0)$. During the out-of-control period, T_i/σ^2_1 is distributed as a non-central chi-square distribution with m degrees of freedom and a non-centrality parameter λ_1; being $\lambda_1 = m(\delta + d)^2$ if $\xi_i = d$, otherwise $\lambda_1 = m(\delta - d)^2/\gamma^2$. If devices such as go and no-go gauges can be used, then measurements are not required except when the sampling goes to the second stage. The proposed sampling procedure is highly recommended for processes where such devices can be employed. During the in-control period, the rate of inspected items per sampling, $\bar{n}$, is given by:

$$\bar{n} = 1 + n_0[2\Phi(-w)] \tag{2}$$

If the parameters n_0 and w are designed to make $\bar{n}$ equal to n, the size of the samples when the joint $\bar{X}$ and R charts are in use, then the joint charts and the DS χ^2 chart will demand the same average number of items (ANI) to be inspected.

Let Q be the probability of deciding that the process is out of control:

$$Q = \Pr\left[(|X_{i1} - \mu_0| > w\sigma_0) \cap (T_i > k\sigma_0^2)\right] \tag{3}$$

The effectiveness of a control chart in detecting a process change can be measured by the average run length (ARL), which is the expected number of samples drawn until the chart gives a signal. The number of samples drawn until a signal is a geometrically distributed random variable with parameter Q. Usually, the process starts in control and some time in the future an assignable cause shifts the process mean and/or increases the process variance. This assumption was assumed for the developed model. When a process is in control, it is desirable that the average number of samples taken since the beginning of monitoring until a signal (ARL_0) be large; this guarantees few false alarms. The $ARL_0 = \alpha^{-1}$, where α is the type I error probability. The ARL_0 was chosen to be 433.0 (the same value adopted by Costa and Rahim, 2004). When a process is out of control, it is desirable that the average number of samples taken since the occurrence of the assignable cause until a signal (ARL) be small, this guarantees fast detection of process changes. The

ARL = $(1 - \beta)^{-1}$, being β the type II error probability. In the Appendix, we show how the Equation (3) can be used to obtain the false alarm risk (α) and the power $(1 - \beta)$ of the *DS* χ^2 chart.

Tables 2 and 3 provide the *ARL* for the *DS* χ^2 chart and for the joint $\bar{X}$ and *R* charts. One can see from these tables that, in most of the cases, the *DS* χ^2 chart always detects process changes faster (lower *ARL*) than the joint $\bar{X}$ and *R* charts. The exceptions occur in some cases when there is a large change in the mean (boldfaced values). For given $\bar{n}$, *m* and *d*, the *ARL* for the joint $\bar{X}$ and *R* charts and *DS* χ^2 chart decreases as δ and/or γ increases. One can see from both tables that the *ARL* value always decreases as $\bar{n}$ increases.

Table 2 – Values of the ARL for the joint $\bar{X}$ and *R* charts and for the DS χ^2 chart ($\bar{n} = n = 3$).

			$m = 6$, $w = 0.8416$			$m = 9$, $w = 1.15035$			$m = 12$, $w = 1.3352$		
		D=	0.5	0.7	0.9	0.5	0.7	0.9	0.5	0.7	0.9
		K=	25.1365	28.9010	33.3133	30.6900	35.4754	41.2003	35.8120	41.5380	48.4800
γ	δ	$\bar{X}$ – *R**									
	0	433.0	433.0	433.0	433.0	433.0	433.0	433.0	433.0	433.0	433.0
	0.50	102.8	46.40	43.10	40.05	35.30	30.67	28.08	28.77	24.74	22.51
1.0	0.75	37.51	15.23	13.59	12.65	10.61	9.38	8.68	8.72	7.75	7.20
	1.0	15.25	6.04	5.50	5.18	4.38	4.02	3.81	3.88	3.62	3.50
	1.25	7.14	3.04	2.83	2.71	2.43	2.32	2.25	2.38	2.32	2.28
	1.50	3.89	1.89	1.81	1.76	1.71	1.68	1.66	1.80	1.79	1.79
	2.00	1.71	1.20	1.19	1.18	1.25	1.25	1.25	1.34	1.34	1.34
	0	33.46	24.01	26.76	29.35	19.64	22.33	25.04	17.03	19.58	22.23
	0.50	19.53	10.75	10.92	11.11	6.48	8.61	6.77	7.39	7.52	7.66
1.3	0.75	11.98	6.06	5.99	5.97	4.84	4.79	4.78	4.38	4.35	4.34
	1.0	7.30	3.68	3.61	3.57	3.08	3.03	3.01	2.93	2.91	2.90
	1.25	4.63	2.47	2.43	2.40	2.20	2.18	2.17	2.22	2.21	2.21
	1.50	3.12	1.83	1.81	1.79	1.75	1.74	1.74	1.83	1.83	1.84
	2.00	1.75	1.29	1.29	1.28	1.35	1.35	1.35	1.43	1.43	1.44
	0	13.05	8.89	10.02	11.16	7.20	8.21	9.29	6.32	7.20	8.19
	0.50	9.61	5.65	6.22	6.56	4.61	5.13	5.44	4.36	4.65	4.94
1.5	0.75	7.09	4.12	4.25	4.37	3.48	3.60	3.71	3.27	3.39	3.49
	1.0	5.09	2.95	2.99	3.03	2.60	2.65	2.69	2.54	2.60	2.65
	1.25	3.68	2.22	2.24	2.25	2.06	2.08	2.10	2.09	2.12	2.15
	1.50	2.75	1.78	1.78	1.79	1.73	1.74	1.76	1.80	1.82	1.84
	2.00	1.73	1.33	1.33	1.34	1.38	1.39	1.40	1.47	1.47	1.48
	0	3.78	2.72	2.96	3.24	2.42	2.60	2.82	2.33	2.47	2.64
	0.50	3.43	2.46	2.65	2.85	2.23	2.37	2.54	2.20	2.30	2.44
2.0	0.75	3.09	2.22	2.36	2.50	2.06	2.16	2.29	2.06	2.14	2.24
	1.0	2.71	1.98	2.07	2.17	1.89	1.96	2.04	1.92	1.97	2.05
	1.25	2.36	1.77	1.83	1.90	1.73	1.78	1.84	1.78	1.82	1.87
	1.50	2.06	1.59	1.63	1.68	1.60	1.63	1.67	1.67	1.69	1.73
	2.00	1.60	1.35	1.37	1.39	1.41	1.42	1.44	1.48	1.49	1.50

*$\bar{X}$ chart with control limits $\mu_0 \pm 3.250\,\sigma_0/\sqrt{3}$, and R chart with upper control limit 5.009 σ_0.

Table 3 – Values of the ARL for the joint $\overline{X}$ and R charts and for the DS χ^2 chart ($\bar{n} = n = 5$).

			$m = 10$ $w = 0.7665$			$m = 15$ $w = 1.0676$			$m = 20$ $w = 1.2521$		
*		D =	0.5	0.7	0.9	0.5	0.7	0.9	0.3	0.5	0.7
		K =	33.3743	38.6042	44.8822	41.9790	48.7906	57.1134	43.9203	50.0360	58.3310
γ	δ	$\overline{X}-R$*									
	0	433.0	433.0	433.0	433.0	433.0	433.0	433.0	433.0	433.0	433.0
	0.50	56.59	28.94	24.74	22.42	20.66	17.38	15.62	22.70	16.82	14.15
1.0	0.75	16.94	8.26	7.21	6.62	6.02	5.31	4.92	6.38	5.26	4.74
	1.0	6.39	3.37	3.07	2.90	2.79	2.62	2.53	3.00	2.78	2.69
	1.25	3.07	1.92	1.82	1.77	1.86	1.82	1.80	2.05	2.03	2.02
	1.50	1.84	1.41	1.38	1.37	1.51	1.51	1.50	1.67	1.67	1.67
	2.00	1.12	1.12	1.12	1.12	**1.21**	**1.21**	**1.21**	**1.29**	**1.29**	**1.29**
	0	26.24	16.53	19.09	21.69	13.13	15.42	17.88	9.66	11.25	13.28
	0.50	13.23	6.81	6.92	7.05	5.50	5.60	5.70	4.89	4.99	5.11
1.3	0.75	7.33	3.85	3.81	3.79	3.31	3.30	3.29	3.20	3.22	3.24
	1.0	4.20	2.46	2.43	2.41	2.31	2.31	2.30	2.37	2.40	2.42
	1.25	2.63	1.80	1.79	1.78	1.83	1.84	1.84	1.95	1.97	2.00
	1.50	1.84	1.47	1.47	1.47	1.57	1.58	1.59	1.70	1.71	1.72
	2.00	1.20	1.20	1.20	**1.21**	**1.30**	**1.30**	**1.31**	**1.38**	**1.38**	**1.38**
	0	9.46	5.85	6.77	7.77	4.77	5.50	6.36	3.85	4.28	4.88
	0.50	6.62	3.87	4.16	4.43	3.34	3.59	3.83	3.00	3.19	3.43
1.5	0.75	4.67	2.80	2.92	3.01	2.56	2.68	2.78	2.48	2.58	2.71
	1.0	3.25	2.11	2.17	2.21	2.07	2.13	2.18	2.11	2.17	2.25
	1.25	2.33	1.70	1.74	1.76	1.76	1.80	1.84	1.86	1.89	1.94
	1.50	1.77	1.46	1.49	1.50	1.57	1.59	1.62	1.68	1.69	1.71
	2.00	1.24	1.23	1.24	1.25	1.33	1.34	1.35	**1.42**	**1.42**	**1.42**
	0	2.64	1.97	2.13	2.33	1.90	2.00	2.14	1.94	1.97	2.03
	0.50	2.41	1.83	1.96	2.11	1.81	1.90	2.01	1.87	1.90	1.95
2.0	0.75	2.17	1.70	1.80	1.92	1.73	1.79	1.88	1.81	1.83	1.87
	1.0	1.92	1.58	1.65	1.73	1.64	1.69	1.76	1.74	1.75	1.78
	1.25	1.69	1.46	1.51	1.58	1.55	1.59	1.64	1.66	1.66	1.68
	1.50	1.50	1.37	1.41	1.45	1.48	1.50	**1.53**	**1.58**	**1.58**	**1.59**
	2.00	1.23	1.25	1.26	**1.28**	**1.35**	**1.36**	**1.37**	**1.43**	**1.43**	**1.44**

* $\overline{X}$ chart with control limits $\mu_0 \pm 3.250\,\sigma_0/\sqrt{5}$, and R chart with upper control limit 5.432 σ_0.

It can be observed from Table 2 that for large disturbances in the mean ($\delta = 2.0$ and in some cases $\delta = 1.5$), the *ARL* value increases as m increases. The choice of d affects the speed with which the DS χ^2 chart signals. In general, larger values of d are better for detecting shifts in μ with $\sigma = \sigma_0$, and worse for detecting increases in σ with $\mu = \mu_0$. For example, one can see from Table 3, when $\delta = 0.5$, $\gamma = 1.0$, as d increases, the *ARL* value decreases from 28.94 to 24.74 to 22.42, for $m = 10$; from 20.66 to 17.38 to 15.62, for $m = 15$; from 22.70 to 16.82 to 14.15, for $m = 20$. On the other hand, when $\delta = 0$, $\gamma = 1.5$, as d increases, the *ARL* value increases from 5.85 to 6.77 to 7.77, for $m = 10$; from 4.77 to 5.50 to 6.36, for $m = 15$; from 3.85 to 4.28 to 4.88, for $m = 20$.

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The performance of a control chart can also be measured by the sampling rate. When the joint $\bar{X}$ and R charts are in use, the sampling rate is given by n (the sample size), and when the $DS\ \chi^2$ chart is in use, the sampling rate is given by $\bar{n}$ (see Equation 2). Like the *ARL*, the sampling rate can be used to compare two charts. To keep the comparison meaningful, the two charts should offer the same protection against false alarms by having the same ARL_0. For specified types of process changes, the control chart with the best performance is the one with the lowest sampling rate and best ability of detection. One can see from Table 4 that the $DS\ \chi^2$ chart is an interesting alternative to the joint $\bar{X}$ and R charts, if the aim is the reduction of the sampling rate. For example, one can see from Table 4, when $n = 3.0$, $\delta = 1.0$, and $\gamma = 1.3$, the $DS\ \chi^2$ chart leads to a better trade-off between the time to detect the process change and the sampling rate required to control the process, that is lower *ARL* (5.01 against 7.30), and lower *Sampling Rate* (1.5 against 3.0).

The Design of the Non-Central Chi-Square Chart with Double Sampling

The use of the $DS\ \chi^2$ chart requires the specification of d, n_0, $\bar{n}$, and k. According to Equation (2), w is a function of $\bar{n}$ and n_0. If the practitioners have some idea about the

Table 4 - Values of the ARL for the joint $\bar{X}$ and R charts and for the $DS\ \chi^2$ chart ($n \neq \bar{n}$).

γ	δ	$\bar{X}-R$ $n = 3.0$	$DS\ \chi^2$ chart $\bar{n} = 1.5; m = 7; d = 0.25$ $w = 1.7317; k = 21.8140$	$\bar{X}-R$ $n = 5.0$	$DS\ \chi^2$ chart $\bar{n} = 3.0; m = 11; d = 0.30$ $w = 1.2816; k = 30.0043$
1.0	0	433.0	433.0	433.0	433.0
	0.50	102.8	67.58	56.59	39.12
	0.75	37.51	22.43	16.94	11.31
	1.0	15.25	9.12	6.39	4.56
	1.25	7.14	4.62	3.07	2.53
	1.50	3.88	2.87	1.84	1.80
1.3	0	33.46	23.90	26.24	15.59
	0.50	19.53	12.85	13.23	7.64
	0.75	11.98	7.85	7.33	4.60
	1.0	7.30	5.01	4.20	3.02
	1.25	4.63	3.45	2.63	2.22
	1.50	3.12	2.57	1.84	1.80
1.5	0	13.05	9.41	9.46	5.85
	0.50	9.61	6.86	6.62	4.23
	0.75	7.09	5.15	4.67	3.22
	1.0	5.09	3.86	3.25	2.51
	1.25	3.68	2.99	2.33	2.05
2.0	0	3.78	3.28	2.64	2.25
	0.50	3.43	3.04	2.41	2.12
	0.75	3.09	2.79	2.17	1.99
	1.0	2.71	2.54	1.92	1.86
	1.25	2.36	2.29	1.69	1.69
	1.50	2.06	2.06	1.50	1.51

disturbances the process is subject to, then they can use Tables 5 and 6 for selecting the design parameters. These tables provide the values of d, n_0, and k that minimize the average number of samples required to signal δ standard deviation shifts in the process mean accompanied by $100(\gamma^2 - 1)\%$ of process variance increase under the constraints that $\alpha \approx 0.0023$, $\bar{n} = 3$, $3 < n_0 \leq 15$, and $d \leq 1$ (Table 5), or $\bar{n} = 5$, $5 < n_0 \leq 20$, and $d \leq 1$ (Table 6). For example, Table 6 shows that shifts in the process mean equal to half standard deviation ($\delta = 0.5$), accompanied by 56.25% process variance increase ($\gamma = 1.25$) are detected faster when $d = 0.6$, $n_0 = 20$, and $k = 55.6274$.

Comparing Charts

It seems reasonable to compare the *DS* χ^2 chart and the two-stage sampling (*TSS*) $\bar{X}$ and *R* charts in terms of the speed with which they detect process disturbances. When the *TSS* $\bar{X}$ and *R* charts are in use, samples of size $m = n_0 + 1$ are taken from the process at regular

Table 5 – The ARL and the optimum design parameters for the *DS* χ^2 charts, $\bar{n} = n = 3$.

				δ = 0			δ = 0.5			δ = 1.0		
				γ			γ			γ		
n_0	w	d	k	1.00	1.25	1.50	1.00	1.25	1.50	1.00	1.25	1.50
15		0	27.580	433	**16.8**	**4.83**	66.2	8.99	3.88	6.01	3.47	2.62
15		1	50.649	433	29.6	7.78	**18.2**	7.91	4.79	3.52	3.07	2.77
15	1.501	0.6	36.628	433	22.8	6.05	22.1	**7.83**	4.26	3.70	3.09	2.66
15		0.1	27.855	433	17.0	4.87	48.8	8.42	**3.84**	5.02	3.28	2.56
13	1.423	1	45.936	433	31.1	8.18	19.6	8.25	4.89	**3.43**	2.99	2.70
10	1.282	1	38.667	433	34.01	9.03	23.2	9.15	5.20	3.47	**2.97**	2.66
10		0.3	23.424	433	22.2	5.85	39.1	9.28	4.23	4.56	3.19	**2.51**
Standard $\bar{X}$ and *R* charts				433	45.40	13.10	103.00	23.80	9.61	15.30	8.08	5.09

Boldfaced ARL is the minimum one for the specified δ and γ (for which the minimization was performed).

Table 6 – The ARL and the optimum design parameters for the *DS* χ^2 charts, $\bar{n} = n = 5$.

				δ = 0			δ = 0.5			δ = 1.0		
				γ			γ			γ		
n_0	w	d	k	1.00	1.25	1.50	1.00	1.25	1.50	1.00	1.25	1.50
20		0	40.8006	433	**12.1**	**3.57**	52.6	6.43	2.94	4.23	2.65	2.14
20		1	76.8562	433	23.5	5.90	**11.9**	5.73	3.74	2.67	2.49	2.36
20	1.282	0.6	55.6274	433	17.3	4.49	14.8	**5.68**	3.30	2.76	2.50	2.23
20		0.1	41.5770	433	12.3	3.59	37.0	6.02	**2.92**	3.56	2.54	2.12
15	1.111	1	64.3750	433	26.5	6.62	14.3	6.27	3.88	**2.50**	**2.34**	2.21
15		0.2	36.4343	433	14.9	3.94	32.9	6.51	3.02	3.39	2.46	**2.02**
Standard $\bar{X}$ and *R* charts				433†	36.7	9.50	56.6	16.2	6.62	6.39	4.48	3.25

Boldfaced ARL is the minimum one for the specified δ and γ (for which the minimization was performed).

time intervals. The sampling is performed in two stages. During the first stage, the first item of the sample is inspected. If its X value is close to the target value ($|X - \mu_0| < w\sigma_0$, $w > 0$), then the sampling is interrupted. Otherwise, the sampling goes on to the second stage, where the remaining n_0 items are inspected and the $\bar{X}$ and R values are computed. The signal is given by an $\bar{X}$ value beyond the control limits ($\mu_0 \pm k_{\bar{X}}\sigma_0/\sqrt{n_0}$) and/or by an R value above the upper control limit, $k_R(n_0)\sigma_0$. The sample values can be computed taking into account all m values (*DS* procedure), or only the remaining n_0 values (*TSS* procedure). The *DS* charts are more effective in detecting process disturbances. This result is intuitive, once the *DS* scheme makes better use of the sample information. Table 7 provides the *ARL* for the *DS* χ^2 chart and for the *TSS* $\bar{X}$ and R charts. One can see from this table that the *chi-square* chart competes with the joint charts. Thus, a single chart can be used for monitoring both the process mean and variance.

Very recently, Reynolds and Stoumbos (2004) investigated the use of two *EWMA* charts or two CUSUM charts for monitoring the mean and the variance. In their study, the $EWMA_X$ chart for detecting changes in μ is based on the control statistic

$$E_k^X = (1-\lambda)E_{k-1}^X + \lambda\bar{X}_k,\ k = 1, 2, \dots, \tag{4}$$

where: λ is a tuning parameter satisfying $0 < \lambda \leq 1$ and the starting value is usually taken to be $E_0^X = \mu_0$. A signal is given at sample k if E_k^X falls outside the control limits

$$\mu_0 \pm h_{EX}\sqrt{\lambda/(n(2-\lambda))}\,\sigma_0 \tag{5}$$

where: $\sqrt{\lambda/(n(2-\lambda))}\,\sigma_0$ is the asymptotic in-control standard deviation of E_k^X. The $EWMA_{X^2}$ chart for detecting changes in σ is based on the control statistic

$$E_k^{X^2} = (1-\lambda)\max\left\{E_{k-1}^{X^2}, \sigma_0^2\right\} + \lambda\sum_{i=1}^{n}\frac{(X_{ki}-\mu_0)^2}{n},\ k = 1, 2, \dots, \tag{6}$$

where: $E_0^{X^2} = \sigma_0^2$. A signal is given at sample k if $E_0^{X^2}$ falls above the control limit

$$\sigma_0^2 + h_{EX^2}\sqrt{2\lambda/(n(2-\lambda))}\,\sigma_0^2 \tag{7}$$

Table 7 – Values of the *ARL* for the *TSS* $\bar{X}$ and *R* Charts and for the *DS* χ^2 chart ($\bar{n} = n = 5$; $n_0 = 14$; $w = 1.068$).

	TSS $\bar{X}$ *and R charts** $k_{\bar{X}} = 2.873$; $k_R(n_0) = 5.725$							*DS* χ^2 chart $k = 42.6115$; $d = 0.7$					
	δ							δ					
γ	0.00	0.50	0.75	1.00	1.25	1.50	γ	0.00	0.50	0.75	1.00	1.25	1.50
1.0	433.0	18.01	5.13	2.51	1.78	1.49	1.0	433.0	17.38	5.17	2.62	1.82	1.51
1.3	19.29	7.50	3.88	2.42	1.82	1.55	1.3	15.42	5.60	3.30	2.31	1.84	1.58
1.5	6.57	4.33	3.02	2.22	1.79	1.56	1.5	5.50	3.59	2.68	2.13	1.80	1.59
2.0	2.13	1.99	1.85	1.71	1.59	1.49	2.0	2.00	1.90	1.79	1.69	1.59	1.50

**ARL* values tabulated in Costa and Rahim (2004).

The CUSUM$_X$ chart for detecting changes in μ is based on two separate one-side CUSUM statistics. The upper CUSUM statistic for detecting increases in μ is

$$C_k^{X+} = \max\left\{0, C_{k-1}^{X+}\right\} + \left(\overline{X}_k - \mu_0 - \zeta_\mu \sigma_0/2\right), k = 1, 2, ..., \tag{8}$$

and the lower CUSUM statistic for detecting decreases in μ is:

$$C_k^{X-} = \min\left\{0, C_{k-1}^{X-}\right\} + \left(\overline{X}_k - \mu_0 + \zeta_\mu \sigma_0/2\right), k = 1, 2, ..., \tag{9}$$

The initial values for the statistics are $C_0^{X+} = C_0^{X-} = 0$. The chart parameter ζ_μ is defined as $\zeta_\mu = |\mu_1 - \mu_2|/\sigma$, where μ_1 is an out-of-control value of μ that should be detected quickly.

The chart signals if:

$$C_k^{X+} > h_{CX}\sigma_0/\sqrt{n} \text{ or } C_k^{X-} < -h_{CX}\sigma_0/\sqrt{n} \tag{10}$$

The CUSUM$_{X^2}$ chart for detecting changes in σ is based on the control statistic:

$$C_k^{X^2} = \max\left\{0, C_{k-1}^{X^2}\right\} + \left(\sum\frac{(X_{ki} - \mu_0)^2}{n} - \frac{21n\zeta_\sigma}{1 - \zeta_\sigma^{-2}}\sigma_0^2\right), \qquad k = 1, 2, ..., \tag{11}$$

where $C^{X^2}{}_0 = 0$ and the upper control limit is

$$h_{CX^2}\sigma_0^2/n \tag{12}$$

The chart parameter ζ_σ is defined as $\zeta_\sigma = \sigma_1/\sigma_0$, where σ_1 ($\sigma_1 > \sigma_0$) is a value of σ that should be detected quickly.

Comparing the *EWMA* and the CUSUM chart combinations with the corresponding *DS Chi-square* chart in Table 8 shows that the *EWMA* and the CUSUM chart combinations are better for small shifts in μ, but the *DS* χ^2 chart is equivalent or slightly better for changes in σ ranging from $\gamma = 1.25$ to $\gamma = 3.00$.

Illustrative Example

The joint $\overline{X}$ and *R* charts have been used to monitor the diameter of shafts. As the specifications of the diameters (0.7500 ± 0.0030 inches) are very tight, a minor shift in the process mean accompanied by increases in the variance leads to the manufacturing of a large number of shafts with diameters beyond the specifications. Past data show that the standard deviation (σ) of the diameters, originally stable at $\sigma_0 = 0.0012$ inches, increases when the process mean goes off-target. The parameters of the joint charts are $n = 3$, $k_{\overline{X}} = 3.250$ and $k_R(n) = 5.009$. The *Double Sampling Non-Central Chi-Square* Chart was designed to replace the joint charts. At the first stage, one item of the sample is inspected. If its X value belongs to the interval $\mu_0 \pm w\sigma_0$ ($w = 1.3352$), then the sampling is interrupted. Otherwise, at the second stage, the remaining sample items are inspected, $T = \sum\left(X_j - \mu_0 + \xi\sigma_0\right)^2$ is computed and its value is plotted on the *Non-Central Chi-Square Chart* with $k = 41.5380$ ($m = 12$; $d = 0.7$).

Table 8 – *ARL* for the EWMA and CUSUM chart combinations and for the *DS* χ^2 chart.

$\bar{n} = n = 4$		$EWMA_{\bar{X}}$ and $EWMA_{X^2}$ $\lambda = 0.1000$ $h_{E\bar{X}} = 2.940$ $h_{EX^2} = 3.412$	$CUSUM_{\bar{X}}$ and $CUSUM_{X^2}$ $\zeta_\mu = 0.8$ $\zeta_\sigma = 1.35$ $h_{C\bar{X}} = 6.772$ $h_{CX^2} = 19.143$	*DS* χ^2 chart $n_0 = 16$ $d = 0.40$ $w = 1.3180$ $k = 41.0710$
δ	γ			
0	1.00	370.4	370.4	370.4
0.25	1.00	33.1	33.1	107.4
0.50	1.00	10.6	10.4	21.4
1.00	1.00	4.3	4.2	3.2
1.50	1.00	2.6	2.6	1.7
2.00	1.00	1.7	1.8	1.3
3.00	1.00	1.1	1.1	1.1
0	1.25	16.1	16.2	16.2
0	1.50	5.9	5.9	4.5
0	2.00	2.6	2.6	2.1
0	2.50	1.8	1.8	1.7
0	3.00	1.4	1.4	1.4

From Table 2, one can see that the *DS* χ^2 chart detects process disturbances faster than the joint $\bar{X}$ and *R* charts. For instance, to detect mean shifts of half standard deviation ($\delta = 0.5$) accompanied by 69% of process variance increase ($\gamma = 1.3$), the *DS* χ^2 chart, with $n_0 = 11$ and $d = 0.7$, requires, on average, 7.52 samples ($ARL = 7.52$), against 19.53 samples (the *ARL* value for the joint $\bar{X}$ and *R* charts with $n = 3$, $k_{\bar{X}} = 3.250$ and $k_R(n) = 5.009$).

During the in-control period, approximately 82% of the two stage procedure does not go to the second stage (that is, $Pr[\,|Z| < w = 1.335\,|\,Z \approx N(0,1)] \approx 0.82$); consequently, most of the time the user will not have to measure the diameter of any shaft. As the process is very stable (remains in control most of the time) the occurrence of $\bar{X}$ or *R* values outside the control limits is rare. The activity of measuring three shafts at each hour plus computation of $\bar{X}$ and *R* values may be considered tedious. The *DS* procedure provides some relief once the average length of the interval between no-interrupted samplings (when measurements are effectively required) is longer than 5 hours and half (1/0.18 sampling intervals of one hour). Hence, most of the time, measurements will be performed once or twice per day (considering 8 hours of labor). One can raise the question that it is more monotonous to deal with the non-central chi-square statistic than with $\bar{X}$ and *R* statistics, however, with a programmable calculator, the required number of keystrokes to obtain the *T* values or the $\bar{X}$ values is just the same. In summary, the *DS* χ^2 chart is not only more sensitive than the joint $\bar{X}$ and *R* charts, but operationally simpler as well.

Conclusions

In this paper we have shown that it is possible to design one chart, which can monitor both the process mean and variance. The *DS* χ^2 control chart is of particular interest when the quality characteristic can be evaluated either by attribute or variable. As it is much easier to deal with attributes than with variables the proposed chart is a valuable tool. A classical example consists in monitoring the diameter of shafts. During the first stage of the sampling, a gauge is used to decide if the whole sample must be inspected by variable (for instance, using a micrometer) or if the sampling must be interrupted. This two-stage procedure has two advantages. The first advantage is the reduced number of times the user will need to perform measurements, and the second is the gain in speed with which process disturbances will be detected. The *DS* χ^2 chart was conceived to be a practical tool for surveillance of processes subject to small to moderate disturbances. Moreover, when the process is stable, and the joint $\bar{X}$ and *R* charts are in use, the monitoring becomes monotonous because an $\bar{X}$ or a *R* values rarely falls outside the control limits. The natural consequence is that the user pays less and less attention to the steps required to obtain the $\bar{X}$ and *R* value. But in some cases, this lack of attention can result in serious mistakes. When the *DS* χ^2 chart is in use, most of the samplings are interrupted, consequently, most of the time the user will be working with attributes. Our experience shows that the inspection of one item by attribute is much less monotonous than measuring three, four or five items at each sampling.

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Appendix: Computation of the Risk α and the Power of the Control Chart 1-β.

The statistic T is given by:

$$T = \sum_{j=1}^{m}\left(X_j - \mu_0 + \xi\sigma_0\right)^2 \qquad i = 1,2,...$$

$$= \sum_{j=2}^{m}\left(X_j - \mu_0 + \xi\sigma_0\right)^2 + \left(X_1 - \mu_0 + \xi\sigma_0\right)^2$$

$$= \sum_{j=2}^{m}\left(X_j - \overline{X}\right)^2 + n_0\left(\overline{X} - \mu_0 + \xi\sigma\right)^2 + \left(X_1 - \mu_0 + \xi\sigma_0\right)^2$$

$$= \sum_{j=2}^{m}\left(X_j - \overline{X}\right)^2 + n_0\left(\overline{X} - \mu_0 - \delta\sigma_0 + \delta\sigma_0 + \xi\sigma_0\right)^2 + \left(X_1 - \mu_0 - \delta\sigma_0 + \delta\sigma_0 + \xi\sigma_0\right)^2$$

where: $\overline{X} = \sum_{j=2}^{m} X_j$. Dividing T by σ^2_1:

$$T/\sigma_1^2 = \sum_{j=2}^{m}\left(\frac{X_j - \overline{X}}{\sigma_1}\right)^2 + \left(Z + \sqrt{n_0}\,\frac{\delta + \xi}{\gamma}\right)^2 + \left(Z_1 + \frac{\delta + \xi}{\gamma}\right)^2 \qquad (A_1)$$

where: $Z = \sqrt{n_0}\,\dfrac{\overline{X} - \mu_0 - \delta\sigma_0}{\gamma\sigma_0} \sim N\ (0,1)$ and $Z_1 = \dfrac{X_1 - \mu_0 - \delta\sigma_0}{\gamma\sigma_0} \sim N\ (0,1)$. Therefore, during the out-of-control period, the first term on the right side of Equation (A_1) follows a chi-square distribution with m-2 degrees of freedom; the second term follows a non-central chi-square distribution with 1 degree of freedom and non-centrality parameter $\lambda = n_0[(\delta + \xi)/\gamma]^2$. Consequently, the sum of these first and second terms follows a chi-square distribution with n_0 degrees of freedom and non-centrality parameter $\lambda = n_0[(\delta + \xi)/\gamma]^2$. To obtain α and 1-β, we consider the Equation (3)

$$Q = \Pr\left[(|X_1 - \mu_0| > w\sigma_0) \cap (T > k\sigma_0^2)\right] = \Pr\left[\left(\left|Z_1 + \frac{\delta}{\gamma}\right| > \frac{w}{\gamma}\right) \cap \left(\frac{T}{\sigma_1^2} > \frac{k}{\gamma^2}\right)\right]$$

$$= \Pr\left[\frac{T}{\sigma_1^2} > \frac{k}{\gamma^2}\,\middle|\, |Z_1 + \frac{\delta}{\gamma}| > \frac{w}{\gamma}\right]\Pr\left[|Z_1 + \frac{\delta}{\gamma}| > \frac{w}{\gamma}\right] =$$

Then, using Equation A_1, we have:

$$Q = \int_{-\infty}^{(-w-\delta)/\gamma} \Pr\left\{\chi^2_{n_0,\,n_0[(\delta - d)/\gamma]^2} > \left[\frac{k}{\gamma^2} - \left(z + \frac{\delta - d}{\gamma}\right)^2\right]\right\} \frac{e^{-0.5z^2}}{\sqrt{2\pi}}\,dz + \qquad (A_2)$$

$$\int_{(w-\delta)/\gamma}^{\infty} \Pr\left\{\chi^2_{n_0,\,n_0[(\delta + d)/\gamma]^2} > \left[\frac{k}{\gamma^2} - \left(z + \frac{\delta + d}{\gamma}\right)^2\right]\right\} \frac{e^{-0.5z^2}}{\sqrt{2\pi}}\,dz$$

The Equation A_2 gives 1-β. Making $\delta = 0$ and $\gamma = 1$, the Equation A_2 gives the false alarm risk α and then, it can be used to determine the parameter k.