Volume 1, Number 1, 2004, pp. 73-92

An Optimal Production Policy Applied to a Flow-shop Manufacturing System

Oscar S. Silva Filho
Wagner Cezarino
Renato Archer Research Center – CenPRA
Email: oscar.salviano@cenpra.gov.br

Abstract
The problem of selecting an optimal production policy for a make-to-stock process is considered. This process is a particular type of flow-shop manufacturing system whose main characteristic is that the inventory levels are strongly influenced by the fluctuation of demand over the future periods of the planning horizon. In order to guarantee that the demand will be met over such periods, an optimal balance policy, between production and inventory levels, must be determined. As will be discussed ahead, such policy relates smooth production levels to a safety stock rule. In this paper, the production strategy is provided by solving a stochastic production planning problem with chance-constraints on production and inventory variables. A modified stochastic dynamic programming algorithm is used as solution technique for this problem. Finally, a simple case study is proposed to illustrate that such optimal sequential solution can be used to provide long-term plan as well as to help managers to get insights about the use of the firm's aggregated resources.
Keywords: Production planning, optimization, probability, dynamic programming, feedback

Introduction

The problem of selecting an optimal production policy for flow-shop manufacturing systems is the object of attention in this paper. These systems have as a main characteristic their strong dependence on the fluctuation of demand. In a long-term planning horizon, due to the uncertainty of the demand's behavior, the dynamics of flow-shop systems can be understood as a stochastic process. Under this uncertain environment, managers are induced to develop production planning policies that are able to anticipate future fluctuations of demand. This means that managers must find a tradeoff between the maximum customer satisfaction level and the minimum safety-stock level. Besides, another important correlated managerial requirement is to minimize the idleness in the production process capacity.

Volume I, Number I, 2004, pp. 73-92

A stochastic optimization model with constraints on decision variables is used to represent the class of the production planning problem discussed here. Once a stochastic dynamic process is considered, such model can be classified as being an optimal stochastic control problem with chance constraints on state and control variables (Brison and Ho, 1975). Due to the stochastic nature and high dimensionality found in this kind of problem, a true optimal solution is almost impossible to be provided, excepting in very particular cases as discussed in Silva Filho (2000). Looking for alternatives in the literature, it is possible to find different ways to overcome (or, at least, to reduce) such complexity by applying a set of transformations to the original problem, objecting to simplify it (see a brief discussion of these alternatives in section 3.1). These simplifications allow to eliminate, at least partially, the stochastic nature as well as to reduce the dimension of the original problem (Bertesekas, 1995). Statistics hypotheses associated with the randomness of demand as well as the linearity of inventory equation are strategies often used to transform the stochastic problem and, simultaneously, preserve its structural properties. For example, using first and second moments the probabilistic constraints (i.e., the chance-constraints of the stochastic problem) can be transformed into equivalent deterministic constraints (see section 4.1). Finally, a particular advantage of using an equivalent problem is the possibility to apply any appropriate technique of mathematical programming and/or optimal control theory, available in the literature (see Bensoussan et al. (1978) and Bertesekas (1995)).

Since the interest in this study is to investigate long-term planning, all decision variables of the problem are assumed to be found within an aggregated pattern. This means that all similar products – as, for example, those ones sharing the same production line – are considered to belong to a same group (or family) of products. This assumption is very important to choose solution's techniques for production planning because it reduces enormously the dimension of the problem, allowing that well-structured approaches can be applied. Within this context, a state-space optimal control problem can be formulated and, as a result, closed-loop solution (i.e., true optimal solution) by using the Stochastic Dynamic Programming (SDP) algorithm can be provided. In practice, this algorithm can only be applied directly to small dimension problems. This is, of course, the major drawback of the SDP. However, in the literature, it is possible to find large-scale problems, being split into smaller problems that can be easily solved by SDP in a parallel programming pattern (Bertesekas, 1995). It is worth clarifying that parallel programming algorithms will not be object of discussion in this paper.

The sequential optimal solution, provided by SDP, can be used by managers to elaborate a production plan to the company. This solution contains optimal level of aggregate resources required to meet the demand over each period of the planning horizon. In the managerial practice, the production plan is a very important strategy to the company. In fact, the plan is used as a target or goal to be reached by other decisions to be done in the medium and short term planning. Another important advantage of this plan is that it can be used by managers to get insight about the future use of the aggregated material resources of the company.

This paper is an extended version of the paper presented in Silva Filho and Cezarino (1999). The purpose of it is to adapt the traditional SDP algorithm to deal with a class of the stochastic control problem with chance-constraints on decision variables. The modified SDP is named here as Non-conventional SDP algorithm. From now, the acronym used to denote this algorithm will be NSDP.

The paper is distributed as follows: section 2 discusses briefly basic concepts related to the flow-shop nature of the production planning problem. In the section 3, the stochastic production problem is formulated and the notation is introduced. Section 4 and 5 discuss respectively the aspects of transforming the stochastic constraints into deterministic equivalent constraints and the statement of the NSDP algorithm. Lastly, in the section 6, an illustrative example of application is presented. It is a simple example to illustrate important characteristics about the use of the model.

The Production Planning Problem

The manufacture production process can be split into two distinct classes named as flow shop or job shop processes. Table 1 summarizes some aspects related to the nature of these processes.

Table 1 – Characteristic of the manufacturing process.

Process	Nature	Layout	Strategy
Flow Shop	Continuous Dedicated Intermittent	Rigid: Product emphasis	Make to stock
Job Shop	Customer specification	Flexible: Process emphasis	Make to order

It is possible to classify the flow shop process according to its nature as continuous, repetitive and intermittent whose features are discussed in the literature, see for instance, Fogarty et al. (1991). In this kind of environment, the great challenge is to deal with stock policies. Since the flow shop process often uses the same sequence of machines (e.g. assembly lines) to produce different products, it is very important to control both intermediate and final inventory levels. In both cases, the objective is to guarantee customer satisfaction which means a great managerial effort to provide: low cost, high quality and ready delivery. Such objective is essential to preserve the organizational competitiveness within the current globalized market place.

Volume 1, Number 1, 2004, pp. 73-92

The make-to-stock strategy puts enormous emphasis in the use of a set of managerial decisions that increases the company's productivity, leading, as a consequence, to improve the customer satisfaction level. Remember that the customer usually does not tolerate delays in the products delivery. Thus, an important task performed by managers is to develop strategies that are able to anticipate future fluctuation of demand. In fact, they must develop a protective mechanism to meet unexpected demand over the future periods of the planning horizon. For this purpose, it is very important to define a safety-stock rule that, simultaneously, reduces the possibility of the stock out, and does not increase the inventory costs significantly. In order to help managers deal with safety stock rules, a sequential optimal stochastic optimization problem with constraints is formulated in this paper. Since the inventory balance equation represents a dynamic nature of the production process, strategies and techniques from optimal control theory can be applied to solve the stochastic problem. Figure 1 illustrates features of the make-to-stock process and its relation to the stochastic control theory. Some of these features illustrated by figure 1 are: (a) the managerial uncertainty about the demand fluctuation that forces managers to use statistic methods for forecasting future demand from the market; and (b) the feedback of current inventory level that helps managers to adjust production rates to face the orders placed by customers.

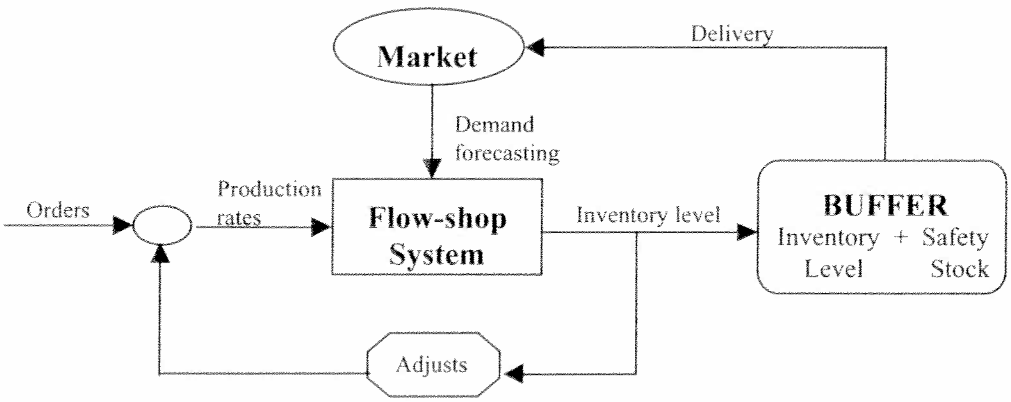


Figure 1 – A make-to-stock environment.

Looking now from a hierarchical decision strategy, the sequential stochastic production planning problem is to be solved over a long-term planning horizon (e.g., from 6 months to one year) (Hax and Candea, 1984). Under this scale of time, the production process is represented by an aggregate dynamic system, for example, as being a unique machine. The process is strongly influenced by the aggregated fluctuation of demand that is a random variable. As a result, both inventory and production variables are random variables. In the practice, these variables are subject to take values from sets formed by lower and upper boundaries. However, the basic question is how to guarantee that such random variables do not violate their physical spaces. In the literature, penalization schemes on the criterion are often used (see Minoux, 1983). In this paper, a chance-constraints scheme will be used in order to preserve them explicitly into the formulation (see discussion in section 4).

Note that this sequential stochastic control problem provides a production planning policy to represent the optimal balance between inventory and production levels. As a consequence, this balance maximizes the customer service level and minimizes the idleness rate of the aggregated production process. As will be seen ahead, these qualitative indexes can be established in advance, setting some probabilistic degrees associated to the inventory and production chance-constraints to appropriate values defined by the manager. For example, setting the probability of inventory chance-constraint equal to 0,95 means that there are 95% of chances of satisfying customers orders with respect to the due-dates previously combined. Thus, this long-term decisions will help the manager to anticipate actions related to the use of company's resources. In parallel, forecasting models of demand can be developed in order to describe its future behavior. During simulation running, this forecasting can be used to analyze different production scenarios.

The Stochastic Model

For each period k of planning horizon $T \geq 1$, the basic notation is given by:

I_k denotes the current inventory level;

P_k denotes the production rate;

D_k is the demand, an independent random variable, with mean $\hat{D}_k$ and variance V_k^D;

$F_k(I_k, P_k)$ denotes the holding and production costs;

I_0 is the initial inventory.

The production-inventory economic balance is described as a linear process given by:

$$I_{k+1} = I_k + P_k - D_k \tag{1}$$

Since the sequence of demands $\{D_k, k=1, 2, ..., T\}$ are, without lost of generality (Silva Filho, 1999), assumed to be independent Gaussian random variables, the balance equation (1) must be understood as being a linear Gauss-Markov stochastic process (Pappoulis, 1991). Assuming that the probability density function of demand is estimated for each period k, and takes into account the linearity of (1), the probability distribution function of inventory can be easily computed. Thus, this distribution function is determined by the sequential evaluation of the first and second statistic moments over the periods of time (i.e., the mean and variance values for each period k) given respectively by the following linear equations:

$$\hat{I}_k = \hat{I}_{k+1} + \hat{P}_k - \hat{D}_k$$
$$V_k^I = V_{k-1}^I + V_k^P + V_k^D \tag{2}$$

It is assumed that the production variable P_k is derived from a closed-loop policy, i.e., like a feedback scheme is illustrated by figure 1. This means that, at each period k, the production variable P_k uses the available information about the inventory variable I_k. For general purpose application, the relation between production and inventory variables can be defined as a non-linear functional feedback structure $\mu_k(.)$. However, in order to assure the linearity and stochastic nature to the balance inventory-production system (1), a *linear feedback gain* is considered and, in this way, the functional structure $\mu_k(.)$ can be stated as follows:

$$P_k = \mu_k(I_k) \equiv -G_k \cdot I_k \tag{3}$$

where G_k denotes the linear gain to be determined in the next section. It is worth mentioning that the linear structure given by (3) will be used to define the spaces of feasible production and inventory variables.

As an immediate consequence of (3), the production variable P_k is also a random variable. Due to a linear dependence between these variables, the production variable is said to be a Gaussian random variable, with mean and variance given, respectively, by $\hat{P}_k$ and V_k^P (Pappoulis, 1991).

With the objective of guaranteeing that both inventory and production random variables do not violate their physical lower and upper boundaries, probabilistic constraints are considered in the formulation. These chance-constraints are expressed as follows:

$$\text{Prob.}\{\underline{I}_k \le I_k < \overline{I}_k\} \ge 2\alpha_k\text{-}1 \tag{4}$$

$$\text{Prob.}\{\underline{P}_k \le P_k < \overline{P}_k\} \ge 2\beta_k\text{-}1 \tag{5}$$

where $(\underline{I}_k, \overline{I}_k)$ and $(\underline{P}_k, \overline{P}_k)$ denote respectively the lower and upper physical capacity limits of inventory and production. The parameters α_k and β_k are probabilistic degrees defined a priori by the manager; they represent the customer service and capacity idleness levels, respectively.

The advantage of the chance-constraints (4) and (5) in relation to the traditional scheme of penalizing such constraints on the criterion is that this strategy allows to preserve the constraints explicitly on the model. Besides, this strategy gives more flexibility to use the model. In fact, managers can vary the probability degrees in order to investigate different types of production scenarios. Thus, it helps managers to improve their knowledge about how to use the production resources rationally.

It is worth observing that whenever the inventory and production lower boundaries in (3) and (4) are greater than zero, an important physical interpretation for the production planning problem can be immediately stated. For example, (a) keeping the lower inventory boundary $\underline{I}_k > 0$ means to get a protection against possible lost due to unexpected situations as, for instance, strikes, excessive demand, machines breakdown. Otherwise, if it is considered that $\underline{I}_k = 0$ means the possibility of backlogging is allowed to occur during the planning; and (b) assuming a production lower boundary $\underline{P}_k > 0$ allows managers to study different policies of production capacity as well as the work-force required to produce the aggregated finished products.

The problem can be formulated as follows: given the initial inventory I_0, a sequential optimal production policy $\{P_0, P_1, \ldots, P_{T-1}\}$ can be obtained by solving the following sequential stochastic production planning problem:

$$\begin{aligned} &\text{Min } E\left\{\sum_{k=0}^{T-1} F_k(I_k, P_k) + F_T(I_T)\right\} \\ &\text{s.t.} \\ &\quad I_{k+1} = I_k + P_k - D_k \\ &\quad \text{Prob.}(\underline{I}_k \le I_k < \bar{I}_k) \ge 2\alpha_k - 1 \\ &\quad \text{Prob.}(\underline{P}_k \le P_k < \bar{P}_k) \ge 2\beta_k - 1 \end{aligned} \tag{6}$$

where the criterion $F(\cdot,\cdot)$ is assumed to be general. It represents a basic structure of the planning costs, based on expected inventory and production costs. Some characteristics as convexity and Gaussian properties can help enormously to handle the mathematical expectation operators (Silva Filho, 1999)

As formulated, the problem (6) is a realistic model for describing a wide class of aggregated production planning problems. In fact, the stochastic nature, the general functional costs, and the constraints, associated with the main decision variables, are the main reasons that lead to such realism. Moreover, the possibility of aggregating an enormous amount of data, available for managerial purposes, allows reducing the dimension of problem significantly. As a consequence, the complexity of the problem is reduced. It is worth mentioning that within a hierarchical decision making scheme, the great part of the well-structured models for planning like (6) are found in the long-term decision levels (i.e., in the strategic level) because of the amount of aggregating available information (see Silva Filho and Ventura, 1999).

Some approaches for solving the problem (6)

Stochastic nature and large dimension are difficulties to be overcome in order to solve (6). Such difficulties impede that a true optimal solution (i.e. a closed-loop solution) can be obtained. Thus sub-optimal solutions are often used in practice. These sub-optimal

Volume I, Number I, 2004, pp. 73-92

approaches are based on strategies that simplify the original stochastic model. Generally, the approximate schemes fall under four different groups briefly described bellow:

a) *Approximation of the functions of the original problem*: non-linearity of the system and non-convexity property of the cost can be substituted, respectively, by a linear system and by a convex function, applying, for example, Taylor series approximation (Bensoussan et al, 1978).

b) *Discretization and Interpolations*: strategies of discretization and interpolation can be used together to reduce the spaces of decision variables and then improve the computer implementation (Bensoussan et al, 1978).

c) *Particular structure for the control law*: a linear rule, for example, can be used to relate the control scheme to current states of the system (Silva Filho and Ventura, 1999).

d) *Informational approximations*: this approximation is related to the amount of information utilized to determine the control law. This implies in reducing the stochastic model into deterministic models with base on the certainty-equivalence principle. There are many different approaches with this characteristic, for example: Naive Feedback Controller and Open-loop Feedback Controller (Silva Filho, 2000).

In the next section, the dynamic programming algorithm is discussed as a technique of solving the problem (6). Some approximate schemes discussed above are used to preserve the main characteristics of the original model. For examples: the linear gain (3) is used as a *particular structure* to guarantee the feasibility space of production and inventory variables. The constraints (4) and (5) are converted in equivalent *deterministic constraints*, preserving the main statistic moments of the stochastic process described by (1).

The Stochastic Dynamic Programming – SDP

The certainty-equivalence principle (Bertesekas, 1995) can be used to develop a sub-optimal strategy to the stochastic problem (6). Such principle states that all random variables of a stochastic problem can be changed by their respective mean values (i.e. their first statistic moments) and, as a consequence, the stochastic problem is converted to an equivalent deterministic problem that is often known as Mean problem. The Mean problem is formulated by setting all random variables of the stochastic problem equal to their mean values (i.e., first statistic moment). This means that the behavior of the demand D_k over the periods will be exactly equal to its average behavior, i.e., $\{D_k = \hat{D}_k, \forall\ 0 \le k \le T\text{-}1\}$. Unfortunately, this kind of solution is not sufficiently reliable to be applied for the company. In fact, this solution does not guarantee that fluctuation of future demand can be met, except if the demand at each period occurs exactly in its expected mean value

Another way of determining a production policy to the stochastic problem (6) is to use the dynamic stochastic programming algorithm (SDP). The direct application of the SDP for

Volume I, Number I, 2004, pp. 73-92

large-scale problems is totally prohibitive because of computational unfeasibility (Bellman, 1957). However, for particular applications where large-scale problems can be decomposed in several small problems, the SDP appears as a good option. The aggregate production planning problems related to make-to-stock environment – where finished products are aggregated in different groups or families – can be included in these particular cases.

In sequel a non-conventional and more realistic approach for developing a production planning policy to the aggregate problem (6) is discussed. Such approach is based on SDP procedure and, thus, it preserves the stochastic features of the problem and maintain explicitly the main statistic moments (i.e., the mean and variance) in both criterion and constraints (4) and (5).

Non-Conventional Stochastic Dynamic Programming (NSDP) Algorithm

The basic idea is to try to guarantee that the solution provided by the SDP approach will not be to violate the inventory and production boundaries (i.e. (4) and (5) constraints). Thus, the first action to define a non-conventional SDP (NSDP) algorithm is to set an equivalent, but deterministic, feasible subspace that reduces the risk associated with the possibility of violation of the constraints (4) and (5). To be created, this subspace takes into account all statistics about the random variable D_k and the linear gain G_k. Next is discussed briefly the three steps to derive this subspace.

(1st step) Transforming the chance-constraint (4) in an equivalent production constraint

$$\mathrm{Prob.}(\underline{I}_k \le I_k < \bar{I}_k) \ge 2\alpha_k - 1 \iff \begin{cases} \mathrm{Prob.}(I_k \le \bar{I}_k) \ge \alpha_k \\ \mathrm{Prob.}(I_k > \underline{I}_k) \ge \alpha_k \end{cases} \tag{7}$$

Using the system equation (1) and linear gain (3), it is possible to write the following expression to inventory level:

$$I_k = [(G_k - 1)/G] \cdot P_k - (\hat{D}_k + \delta_{Dk} \cdot \sigma_{Dk}) \tag{8}$$

where $\sigma_{D_k} = \sqrt{V_k^D}$ is standard deviation of demand for each period k and $\delta_{Dk} \propto N(0,1)$ denotes the residue of demand (i.e., a normal white noise variable). Handling (7) and (8) follows respectively that:

$$\begin{aligned} \mathrm{Prob.}(I_k \le \bar{I}_k) \ge \alpha_k &\iff \mathrm{Prob.}\left(\delta_{D_k} \le \left[((G_k - 1)/G_k) \cdot P_k - \bar{I}_k - \hat{D}_k\right]/\sigma_{D_k}\right) \ge 1 - \alpha_k \\ &\iff P_k \le [G_k/(G_k - 1)] \cdot \left(\bar{I}_k + \hat{D}_k - \sigma_{D_k} \cdot \Phi_{\delta D}^{-1}(\alpha_k)\right) = \bar{P}_k^{\alpha} \end{aligned} \tag{9}$$

$$\begin{aligned} \mathrm{Prob.}(I_k \ge \underline{I}_k) \ge \alpha_k &\iff \mathrm{Prob.}\left(\delta_{D_k} \le \left[((G_k - 1)/G_k) \cdot P_k - \bar{I}_k - \hat{D}_k\right]/\sigma_{D_k}\right) \ge 1 - \alpha_k \\ &\iff P_k \ge [G_k/(G_k - 1)] \cdot \left(\bar{I}_k + \hat{D}_k - \sigma_{D_k} \cdot \Phi_{\delta D}^{-1}(\alpha_k)\right) = \underline{P}_k^{\alpha} \end{aligned} \tag{10}$$

By combining the above (9) and (10) inequalities, it will be created a feasible subspace for the production variable, that is:

$$P_k \in \Omega_\alpha^k = \left\{ P \middle| \underline{P}_\alpha^k \le P_k \le \overline{P}_\alpha^k ; k = 0,1,\dots,T-1 \right\} \tag{11}$$

(2nd step) Transforming the chance constraint (5) in an equivalent inventory constraints

Note that, the production variable is a linear function of inventory variable given by (3); this implies that constraint (5) can be set equal to

$$\Pr ob.(-\underline{P}_k / G_k \le I_k < -\overline{P}_k / G_k) \ge 2\beta_k - 1 \Leftrightarrow \begin{cases} \Pr ob.(I_k \le -\overline{P}_k / G_k) \ge \beta_k \\ \Pr ob.(I_k > -\underline{P}_k / G_k) \ge \beta_k \end{cases} \tag{12}$$

Using an analogous handling scheme applied in the 1st step, follows that:

$$I_k \le ((G-1)/G) \cdot \overline{P}_k - \left(\hat{D}_k + \sigma_{D_k} \cdot \Phi_{\delta D}^{-1}(\beta_k)\right) = \overline{I}_k^\beta \tag{13}$$

$$I_k \ge ((G-1)/G) \cdot \underline{P}_k - \left(\hat{D}_k - \sigma_{D_k} \cdot \Phi_{\delta D}^{-1}(\beta_k)\right) = \underline{I}_k^\beta \tag{14}$$

As a consequence, of combining inequalities (13) and (14), it will result a feasible sub-space for the inventory variable, that is:

$$I_k \in \Omega_\beta^k = \left\{ I \middle| \underline{I}_\beta^k \le I_k \le \overline{I}_\beta^k ; k = 0,1,\dots,T-1 \right\} \tag{15}$$

Note that the spaces provided by (11) and (15) depend strongly on the values chosen for α and β. Based on these values, it is possible to detect *a priori* unfeasible situations (i.e., empty spaces). For example, if, for $\forall k$ and $(\alpha,\beta) \in [1/2, 1)$, occurs that:

$$\begin{aligned} \beta_k &\ge \Phi_{\delta D}\left[\frac{(G_k/(G_k-1)) \cdot (\overline{I}_k - \underline{I}_k)}{2 \cdot \sigma_{D_k}}\right] \Leftrightarrow \Omega_\beta^k \equiv \varnothing \\ \alpha_k &\ge \Phi_{\delta D}\left[\frac{(\overline{I}_k - \underline{I}_k)}{2 \cdot \sigma_{D_k}}\right] \Leftrightarrow \Omega_\alpha^k \equiv \varnothing \end{aligned} \tag{16}$$

Another important characteristic related to sets Ω_α^k and Ω_β^k is that how greater are the values selected for α and β more restrict would become these sets. This means that whenever the manager increases the probabilities of satisfying (4) and (5), the spaces provided by (11) and (15) would become, simultaneously, narrower. For example, let's take two distinct probability measures for the inventory space, i.e., α_1 and α_2 where $\alpha_2 \ge \alpha_1$, as a consequence, it is possible to show that $\Omega_{\alpha_2}^k \subseteq \Omega_{\alpha_1}^k \;\; \forall k$.

(3rd step) Computing the linear gain

The gain G_k provides the necessary adjustments in the production variable P_k in order to maintain the I_k as close as possible of the mean optimal inventory level $\hat{I}_k$ (i.e., $\hat{I}_k$ acts as a desired set-point level to be followed during the planning process). To reduce the impact of variation of inventory and production variables (remember that both variables are Gaussian random variables), the gain is obtained from a minimum variance problem (Astrom, 1970), that is formulated as follows (Geromel and Silva Filho, 1989):

$$\begin{aligned}
&\min_{G_k}\{V_k^I + \lambda_k \cdot V_k^P\} \\
&\text{s.t.} \\
&\quad V_k^I = (1 - G_k)^2 \cdot V_{k-1}^I + V_k^D \\
&\quad V_k^P = G_k^2 \cdot V_k^I
\end{aligned} \tag{17}$$

The linear optimal gain provided by (17) is given by $G_k=1/(1-\lambda_k)$. The parameter λ_k denotes the tradeoff between the simultaneous growing on the evolution of the variances over the periods of planning horizon (Silva Filho, 1999). Note that the problem (17) is used to compute the gain G_k by two reasons. First, it preserves the linear-Gaussian nature of the process; and second, it reduces the variability of the lower and upper boundaries of the spaces Ω_α^k and Ω_β^k, since (17) minimizes the inventory and production variances simultaneously. In fact, note from (11) and (15) that the upper and lower boundaries depend strongly on the demand variances. Following (Geromel and Silva Filho., 1989) the tradeoff parameter can be computed as follows: $\lambda_k = (\Delta I_k)^2/(\Delta P_k)^2$, where $\Delta I_k = \bar{I}_k - \underline{I}_k$ and $\Delta P_k = \bar{P}_k - \underline{P}_k$. This result is due to the application of the Chebyshev theorem (Chou, 1972). See Appendix A to know how to use this theorem to determine the optimal parameter λ_k.

Formulating the NSDP algorithm

From the above results, a non-conventional SDP algorithm applied to the problem (1) can be formulated as follows: an optimal sequential production policy $\{P_0, P_1, ..., P_T\}$ is to be found as solution of the following algorithm:

$$\begin{aligned}
J(I_T) &= F_T(I_T) \\
J_k(I_k) &= \min_{\substack{I_k \in \Omega_\alpha^k \\ P_k \in \Omega_\beta^k}} \mathop{E}_{D_k}\{F_k(I_k) + J_{k+1}(I_{k+1})\}
\end{aligned} \tag{18}$$

The recurrence (18) indicates that it is possible to find a closed-loop policy for the problem (6). This recurrence is implicitly subject to the dynamic evolution of the inventory balance equation (1). The optimal production cost is given by $J^*_{\alpha\beta} = J_0(I_0)$ that depends on the initial inventory level I_0 and the probability measures α_k and β_k provided by the manager. It is worth mentioning that the values chosen for α_k and β_k can provide important managerial insights to managers, particularly, related to their expectation about the customer satisfaction. Therefore, it allows studying policies of meeting demand over the time periods and, at the same time, to investigate the cost/benefits impact of these policies in relation to the productivity rates and customer service. For example, setting $\beta=\alpha=0.95$ indicates respectively that the production operates with almost maximal capacity and the objective is to meet demand at least 95% of time (i.e., delivering finished products on right time, without delays). Surely, this managerial decision has a high impact on the customer service level. Finally, the open-loop solution of (1), which is, a deterministic optimal sequential mean optimal solution for the stochastic problem, is a particular case of NSDP procedure. To verify this, it is enough to set α_k and β_k equal to 1/2.

An Illustrative Example

Consider a hypothetical firm that produces a very large amount of products but very similar. All products can be aggregated into a single family. The managerial objective is to develop an aggregate multi-period optimal production plan for the next 12 months. Such plan must be used by managers to get insight about the rational use of the firm's resources on forwards periods and, therefore, to help them to anticipate managerial decisions, particularly, related to fluctuation of demand.

Problem's data

The main aggregate data are: (a) the lower and upper levels of production capacity are 2 and 10 unites per month, respectively. The lower level is associated with managerial strategies of reducing the idleness level of production capacity; (b) considering that the firm makes to stock, a physical unit of storage of the end-products must be considered. Thus, there are lower and upper limits that cannot be violated by any reasons. For this study, it is assumed an upper limit of storage of 16 unites of end-products and 4 unites of end-product as lower limit (i.e., safety stock). The objective of this safety stock is to be used as prevention against stockout which can lead to loss customers for the concurrence. The sales department maintains a monthly historical of customer orders (i.e. unites of aggregate finished products that are sold per month). From this historical of sales, the analyst can compute the first and second statistic moments (i.e. the mean and variance of demand) that are given by table 2:

Table 2 – First and second statistic moments of demand.

Month	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.
$\hat{D}_k$	7	8	9	7	6	4	5	4	7	7	6	8
Standard deviation: σ_d=2.5												

A quadratic objective function $F(I_k, P_k) = h\cdot(I_k)^2 + c\cdot(P_k)^2$ is used in this example. The production and inventory costs are respectively: c=1.0 e h=2.0. Other costs, such as overtime costs, firing and hiring costs, and so on, will not be considered in this study. The planning horizon is 12 months and initial inventory level is 10 units. The inventory level at the end period (i.e., k=12) is assumed to be free.

Solution

Four cases are discussed now. They are related to the probability measures α and β, chosen a priori by the manager. Using the data above for formulating the problem (6) and, in sequel, applying the NSDP procedure (18) for different probabilities degrees, the four cases were obtained. These results are illustrated in the figures 2 and 3 and a discussion follows next. It is worth mentioning that the space shown in the figure 3 is exactly a composition of the spaces Ω_α^k and Ω_β^k. In this figure, the vertical axis contains the production amplitudes (i.e., the space of production policies given by Ω_α^k) and the horizontal axis contains the lower and upper bounds provided by the space Ω_β^k versus the k periods of planning horizon T.

Case I (α=β=0.5): This means that the plan can fail to meet demand at a given period of the planning horizon. In this case, the solution provided by (18) works well only if the demand is set exactly equal to its average level. Using control theory jargon, one can say that such solution was derived from a system operating in *open-loop* pattern. Figure 2(a) exhibits inventory and production trajectories obtained from a simulation scheme that considers the demand represented by a synthetic random sequence. Note that the lower boundary of inventory variable (i.e. the safety stock level allowed) was violated in two periods (4^{th} and 12^{th} periods). The reason of this is that the "open-loop plan" did not get to anticipate the demand fluctuation, particularly developing a safety stock strategy to prevent against demand oscillations. Thus it is possible to consider that the safety stock, defined by the manager, is not sufficient to meet demand in these periods. As a consequence, there is a high risk of stock out. This undesired occurrence can lead to the loss of customers for the concurrence due to delays in delivering products on right time. It is worth mentioning that to operate under this condition, the manager relies deeply in his persuasion power of negotiating, with his customers, new due-dates of delivering finished products.

Case II (α=β=0.95): In this case, the spaces Ω_α^k and Ω_β^k are extremely more restricted (i.e., narrower) than the case before. See section 4.1 to a brief discussion about the characteristic related to the influence of probabilities degrees over the dimension of the production (11) and inventory (15) spaces. From this case, the characteristic can be visualized by comparing the figures 3(a) and 3(b) that show the combination of the production and inventory spaces in 3D format for α=β=50% and α=β=95%, respectively. Note that whenever managers set the values of α and β near to 1 (one), they intend to improve the customer satisfaction level (i.e., improve the customer service). This means that the main goals are to meet the demand and to minimize the idleness of production capacity, at any period

Volume I, Number I, 2004, pp. 73-92

of planning horizon. Figure 2(b) illustrates this aspect. Differently of case I, in this case, the inventory levels do not violate the lower boundary (dotted line). Evidently, to guarantee this low risk of no-satisfying customer orders, a price must be paid. This price can be measured by the increase of the inventory level at the future periods of planning horizon. As a result, the total production cost increases proportionally. Moreover, comparing the production policies exhibited by figures 2.(a) and 2.(b), it is possible to verify that the production rate, in this case, is smoother than in the case I. It is important to add that the continuous oscillation verified in the production levels (very common in make-to-stock environments) can reveal the weakness of the manager to deal with demand fluctuation by products in future periods of planning horizon.

Case III (α=0.95 β=0.5) and Case IV (α=0.5 β=0.95): The idea here is to consider the values of α and β varying between the extreme values of their probabilistic ranges (i.e., α and $\beta \in [½, 1)$. Setting, for example, α equal to 0.5 and β equal to 0.95, and vice-versa. The results are illustrated in the figures 2(c) and 2(b) that show simulated trajectories, and figures 3(c) and 3(d) that show the spaces of the production policies. Note that the spaces created by the cases III and IV (figure 3(c) and 3(d)) are very close to the spaces created by the case I and II respectively. These characteristics are due to the strong influence of β over the dimension of the space created by (15), as discussed in section 4.1. Note also that small values of β imply in increasing the dimension of space Ω_{α}^{k}. This means that the number of feasible production policies, given by space (11), increases in a measure that the values of α closes to ½. The simulated trajectories illustrated by figures 2(c) and 2(d) follow approximately the same characteristics exhibited by figures 2(a) and 2(b). However, differently of the case I, the inventory trajectory provided by the case III does not violate the lower bound (i.e., the safety stock).

(a) α=β=0.5 and $J_{\alpha\beta}$=215 (b) α=β=0.95 and $J_{\alpha\beta}$=263

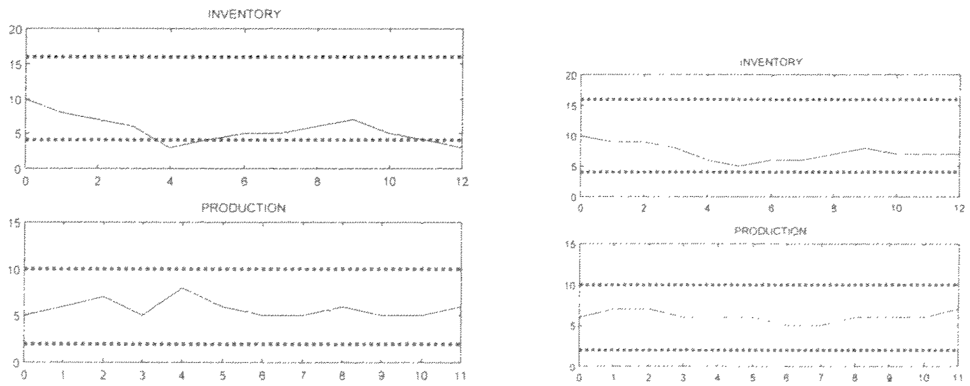


Figure 2 – The simulate optimal inventory and production trajectories.

Volume 1, Number 1, 2004, pp. 73-92

(c) α=0.95, β=0.5 and $J_{\alpha\beta}$=248 (d) α=0.5, β=0.95 and $J_{\alpha\beta}$=252

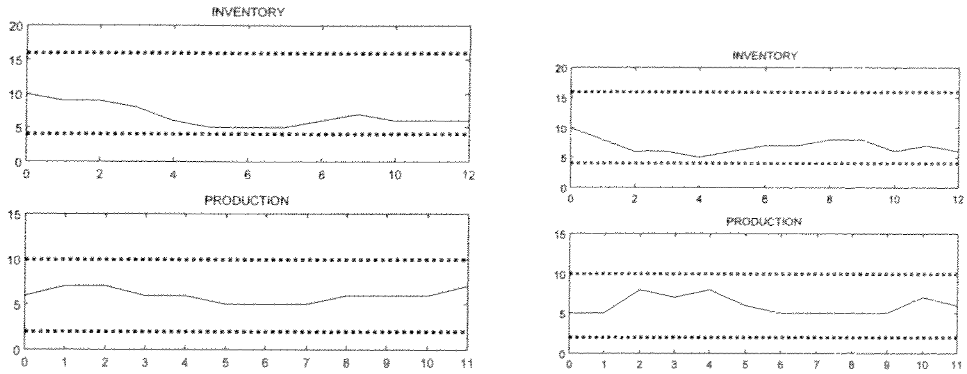
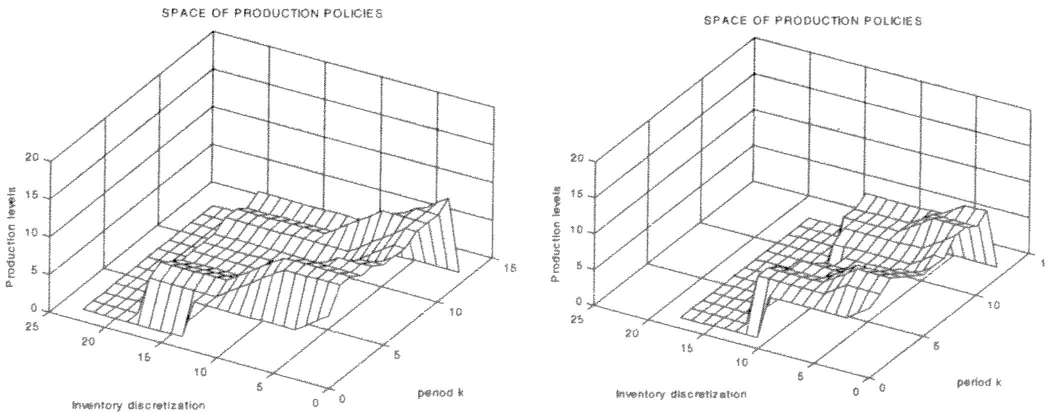


Figure 2 – The simulate optimal inventory and production trajectories – continued.

Comparing the production costs exhibited in the label of figures 2, it is possible to observe that the production policy provided by I (i.e. α=β=0.5) presents the best cost of all cases investigated. On the other hand, the policy provided by the case II (i.e., α=β=0.95) incurred in the worst total cost. Note that the reason for the growing of the cost, in the case II, is the managerial strategy of considering the customer satisfaction level set equal to 95% (i.e., close to 100% of customer service). To reach such satisfaction level, it is necessary to maintain high levels of end-products storage in the warehouses for ready delivery.

(a) α=β=0.5 (b) α=β=0.95



(c) α=0.95, β=0.5 (d) α=0.5, β=0.95

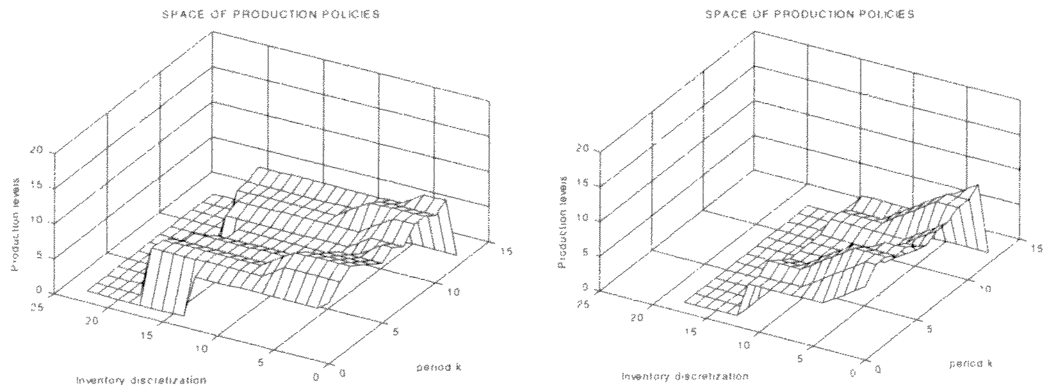


Figure 3 – The space of the production policies.

Conclusions

This paper discussed a way to deal with sequential optimal stochastic production planning problem with chance constraints on decision variables. A constrained stochastic dynamic programming algorithm, denoted as NSDP, was investigated as a possible strategy to develop a production plan. From NSDP, some characteristics related to the equivalent deterministic constraints and the influence of probabilistic degrees in these constraints were introduced. A simple example allowed to visualize these characteristics and to understand the cost/benefit of identifying an appropriate production policy. It is worth mentioning that, in aggregate production planning context, the approach studied here can be used as an important managerial tool. In fact, despite of the dimension, the approach can be used to practical problem related to flow-shop process where the end-products can be aggregated in families. In this case, the large scale problem can be divided in smallest problems and each one can be solved by parallel programming. Finally, the main advantage of this approach is that it can help the manager to obtain insights about the rational use of the firm's resources.

Acknowledgement

This research was supported by CNPq, the Brazilian National Council for Research and Development, under grant No. 500202/2003-6. The authors are indebted to Dr. Fábio Nauras Akhras and reviewers for their valuable comments

References

Astron, K. J. (1970) Introduction to Stochastic Control Theory, Academic Press, New York.

Bellman, R. (1957) Dynamic Programming, Princeton University Press, Princeton, USA.

Bensoussan, A., Kleindorfer, P. R. and Tapiero, Ch. S. (Eds.) (1978) Applied Optimal Control, Studies in the Management Science, TIMS, Vol. 9, North-Holland, Amsterdam.

Bertsekas, D. P.(1995) Dynamic Programming and Stochastic Control, Athena Scientific, Vol. 1, New York.

Bryson, A. R. and Ho, Y (1975) Applied Optimal Control, Hemisphere Publishing Corporation, New York.

Chou, Y. (1972) Probability and Statistics for Decision Making, Holt, Rinehart & Winston, New York.

Fogarty, D. V., Blackstone, J. H., and Hoffmann, T. R. (1991) Production & Inventory Management, 2nd Edition, APICS, South-Western Publishing. 1991.

Geromel, J. C. and Silva Filho, O. S. (1989) Partial Closed-Loop Control Structure for Linear Stochastic System, IEEE Transaction on Automatic Control, Vol. 34, No. 2, pp. 243-246.

Hax, A., and Candea. D.(1994) Production and Inventory Management, Prentice-Hall, New York.

Minoux, M. (1983): Programming Mathematique: Theory et Algorithmes, Dunod, Paris.

Silva Filho, O. S. (2000) Sub-optimal Sequential Strategies for Aggregate Production Planning under Uncertainties, Gestão & Produção, Vol. 7, n.3, pp. 247-268 (in Portuguese).

Silva Filho, O. S. (1999) An Aggregated Production Planning Model with Demand under Uncertainty, Production Planning and Control, Taylor and Francis Ltd., Vol. 10, 8, pp. 745-55.

Silva Filho, O. S. and Ventura S. D. (1999) Optimal Feedback Control Scheme Helping Manager to Adjust Aggregate Industrial Resources, Control Engineering Practices, Vol.7, pp. 555-63.

Silva Filho O. S. and Cezarino, W. (1999) Providing Optimal Production Policies for Make to Stock System, Proceeding of the 14th World IFAC Congress, Vol. A., pp. 249-54.

Papoulis, A. (1991) Probability, Random Variables, and Stochastic Process (Third Edition), McGraw-Hill, New York.

Biography

Oscar S. Silva Filho is currently the leader of the Division of Production Management at Renato Archer Research Center (CenPRA). He was awarded a BSc in Electronic Engineering in 1980 and received his DSc degrees in Automation System in 1989. Dr. Silva Filho has several years of industrial experience and has been consultant in Brazilian manufacturing industries. His current research interest includes the application of stochastic optimal control and mathematical programming techniques in managerial practices as for example inventory control and production planning. His current consulting activities involve the evaluation and design of integrated supply chain management for assembly industries.

Volume I, Number I, 2004, pp. 73-92

Wagner Cezarino, currently works in the Division of Production Management at Renato Archer Research Center (CenPRA). He has been Bachelor in Computer Sciences since 1986. He has several years of experience in the development of applications involving database and tools in the area of production management. He is currently interested in development of applications in Visual Basic, database Access and pages Web using language VB Script.

Appendix A

The idea of this Appendix is to determine the penalization parameter related to the variance problem (17). In this way, it is initially interesting to investigate the effect of shifting the mean value of inventory and production variables into their respective sets $\Omega_{I,\alpha}$ and $\Omega_{P,\beta}$. As these variables are Gaussian random processes, it is almost intuitive that when their mean values shift from the lower boundary to the upper boundary, their density probability functions will be maximum when their mean values reach the center (i.e. the middle) of the interval formed by the lower and upper boundaries of their respective spaces $\Omega_{I,\alpha}$ and $\Omega_{P,\beta}$. This characteristic can be expressed probabilistically as follows:

$$\text{Prob.}(I_k \in \Omega_{I,\alpha}^k \big|_{\forall \hat{I}_k \in \Omega_{I,\alpha}}) \le \text{Prob.}\left(I_k \in \Omega_{I,\alpha}^k \Big|_{\hat{I}_k = \frac{\bar{I}_k + \underline{I}_k}{2}} \right) \tag{A.1}$$

$$\text{Prob.}(P_k \in \Omega_{P,\beta} \big|_{\forall \hat{P}_k \in \Omega_{P,\beta}}) \le \text{Prob.}\left(P_k \in \Omega_{P,\alpha}^k \Big|_{\hat{P}_k = \frac{\bar{P}_k + \underline{P}_k}{2}} \right) \tag{A.2}$$

Let's handle de expression (A.1). First, consider that $\Omega_{I,\alpha}^k = [\underline{I}_k, \bar{I}_k]$ and $I_k = \hat{I}_k + \delta_{I,k}$, where $\hat{I}_k$ is the mean value of inventory and $\delta_{I,k} \sim N(0, V_k^I)$, i.e., a Gaussian white noise. Thus, it is possible to determine that:

$$\begin{aligned}
\text{Prob.}\left(I_k \in \Omega_{I,\alpha}^k \Big|_{\hat{I}_k = \frac{\bar{I}_k + \underline{I}_k}{2}} \right) &= \text{Prob.}\left(\underline{I}_k \le I_k \le \bar{I}_k \Big|_{\hat{I}_k = \frac{\bar{I}_k + \underline{I}_k}{2}} \right) \\
&= \text{Prob.}\left(\underline{I}_k \le \hat{I}_k + \delta_{I,k} \le \bar{I}_k \Big|_{\hat{I}_k = \frac{\bar{I}_k + \underline{I}_k}{2}} \right) \\
&= \text{Prob.}\left(\underline{I}_k - \hat{I}_k \le \delta_{I,k} \le \bar{I}_k - \hat{I}_k \Big|_{\hat{I}_k = \frac{\bar{I}_k + \underline{I}_k}{2}} \right) \\
&= \text{Prob.}\left(-\frac{\bar{I}_k - \underline{I}_k}{2} \le \delta_{I,k} \le \frac{\bar{I}_k - \underline{I}_k}{2} \right) \\
&= \text{Prob.}\left(|\delta_{I,k}| \le \frac{\bar{I}_k - \underline{I}_k}{2} \right)
\end{aligned} \tag{A.3}$$

Besides, considering $\bar{I}_k - \underline{I}_k = \Delta I_k$ and keeping in mind that $\delta_k = I_k - \hat{I}_k$, results that:

$$\Pr ob\left(I_k \in \Omega^k_{I,\alpha} \Big|_{\hat{I}_k = \frac{\bar{I}_k - \underline{I}_k}{2}} \right) = \Pr ob\left(\left| I_k - \hat{I}_k \right| \le \frac{\Delta I_k}{2} \right) \tag{A.4}$$

Applying the Chebyshev's Theorem (see, Chou (1973), pages 60 and 61) it is possible to re-write (A.4) as follows:

$$\Pr ob\left(\left| I_k - \hat{I}_k \right| \le \frac{\Delta I_k}{2} \right) \ge 1 - 4 \cdot \frac{V^I_k}{(\Delta I_k)^2} \tag{A.5}$$

Note that this theorem allows determining how the variance of inventory can be used to provide information about the pattern of probability accumulation in intervals centered on the first static moment (i.e. the mean expected value). As an immediate consequence, comparing the inequalities (A.1) and (A.5) results that:

$$\Pr ob.(I_k \in \Omega^k_{I,\alpha}) = \Pr ob.\left(\underline{I}_k \le I_k \le \bar{I}_k \right) \ge 1 - 4 \cdot \frac{V^I_k}{(\Delta I_k)^2} \tag{A.6}$$

Proceeding, exactly, in analogous way for the distribution of the production variable results that:

$$\Pr ob.(P_k \in \Omega^k_{P,\beta}) = \Pr ob.\left(\underline{P}_k \le P_k \le \hat{P}_k \right) \ge 1 - 4 \cdot \frac{V^P_k}{(\Delta P_k)^2} \tag{A.7}$$

Note that the interest here is to reduce the possibility of unfeasibility of the problem (6). In this way, the idea is to maximize the probability associated with the inventory and production constraint in the problem (6). Thus, observing the inequalities (A.6) and (A.7), this idea can be explored as follows:

$$\text{Max} \ \Pr ob.(\underline{I}_k \le I_k \le \hat{I}_k) + \Pr ob.\left(\underline{P}_k \le P_k \le \hat{P}_k \right) \ge 2 - 4 \cdot \left(\frac{V^I_k}{(\Delta I_k)^2} + \frac{V^P_k}{(\Delta P_k)^2} \right) \tag{A.8}$$

Note also that to maximize the probabilities in inequality (A.8) means to minimize the evolution of the variance present in upper boundary of (A.8). That is:

$$\text{Min}\left(V^I_k + \frac{(\Delta I_k)^2}{(\Delta P_k)^2} \cdot V^P_k \right) \tag{A.9}$$

The problem (A.9) is similar to the problem defined in (17). Comparing both problems, the trade-off parameter λ^* can be determined as being:

$$\lambda^* = \frac{(\Delta I_k)^2}{(\Delta P_k)^2} \tag{A.10}$$

Volume 1, Number 1, 2004, pp. 73-92